

# Cosmological solutions in modified gravity with monomial nonlocality

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## Abstract

We consider cosmological properties of modified gravity with nonlocal term  $R^p \mathcal{F}(\Box) R^q$  in its Lagrangian. Equations of motion are presented. For the flat FLRW metric, and some particular values of natural numbers  $p$  and  $q$  cosmological solutions of the form  $a(t) = Ce^{-\frac{\gamma}{12}t^2}$  are found.

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## 1. Introduction

Modern theory of gravity is general theory of relativity (GR), which was founded by Einstein one hundred years ago and has been successfully confirmed for the Solar System. It is given by the Einstein equations of motion for gravitational field  $g_{\mu\nu}$ :  $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$ , which can be derived from the Einstein-Hilbert action  $S = \frac{1}{16\pi G} \int R\sqrt{-g}d^4x + \int \mathcal{L}_{mat}\sqrt{-g}d^4x$ , where  $g = \det(g_{\mu\nu})$  and units are chosen in such way that  $c = 1$ .

Despite all its successes, GR is not a final theory of gravity. There are many its modifications, which are motivated by quantum gravity, string theory, astrophysics and cosmology (for a review, see [1]). One of very promising directions of research is *nonlocal modified gravity* and its applications to cosmology (as a review, see [2] and [3]). To solve cosmological Big Bang singularity, nonlocal gravity with replacement  $R \rightarrow R + CR\mathcal{F}(\Box)R$  in the Einstein-Hilbert action

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was proposed in [4]. This nonlocal model is further elaborated in the series of  
 15 papers [5, 6, 7, 8, 9, 10].

In this paper we consider the action

$$S_{pq} = \int \left( \frac{R - 2\Lambda}{16\pi G} + R^p \mathcal{F} \left( \frac{\square}{M^2} \right) R^q \right) \sqrt{-g} d^4x \quad (1)$$

where  $R$  is scalar curvature,  $\mathcal{F}(\square) = \sum_{n=0}^{\infty} f_n \square^n$  is an analytic function of the  
 d'Alembert-Beltrami operator  $\square = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$ ,  $g = \det(g_{\mu\nu})$ ,  $M$  is a  
 characteristic scale and  $p$  and  $q$  are natural numbers. For simplicity we take  $M =$   
 1. At the end of this paper we briefly discuss the limit when  $M \rightarrow +\infty$ . In the  
 20 paper [11] action (1) was introduced and constant scalar curvature cosmological  
 solutions were obtained. Also, perturbations around de Sitter background were  
 discussed in [12].

## 2. Equations of motion

Variation of the action (1) with respect to metric yields the equations of  
 motion in the form

$$-\frac{1}{2} g_{\mu\nu} R^p \mathcal{F}(\square) R^q + R_{\mu\nu} W_{pq} - K_{\mu\nu} W_{pq} + \frac{1}{2} \Omega_{pq\mu\nu} = -\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G}, \quad (2)$$

where

$$\begin{aligned} W_{pq} &= p R^{p-1} \mathcal{F}(\square) R^q + q R^{q-1} \mathcal{F}(\square) R^p, \\ K_{\mu\nu} &= \nabla_\mu \nabla_\nu - g_{\mu\nu} \square, \\ \Omega_{pq\mu\nu} &= \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( g_{\mu\nu} \nabla^\lambda \square^l R^p \nabla_\lambda \square^{n-1-l} R^q + g_{\mu\nu} \square^l R^p \square^{n-l} R^q \right. \\ &\quad \left. - 2 \nabla_\mu \square^l R^p \nabla_\nu \square^{n-1-l} R^q \right), \end{aligned} \quad (3)$$

Detailed derivation of the above equations can be found in [11].

In this paper Friedmann-Lemaître-Robertson-Walker (FLRW) metric  $ds^2 =$   
 $-dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$  is used. The signature of a metric is

(1, 3) and the sign of a curvature tensor is chosen such that

$$R_{\mu\nu\alpha}^{\beta} = \partial_{\nu}\Gamma_{\mu\alpha}^{\beta} - \partial_{\mu}\Gamma_{\nu\alpha}^{\beta} + \Gamma_{\mu\alpha}^{\lambda}\Gamma_{\nu\lambda}^{\beta} - \Gamma_{\nu\alpha}^{\lambda}\Gamma_{\mu\lambda}^{\beta}, \quad (4)$$

$$R_{\mu\nu} = R_{\mu\lambda\nu}^{\lambda}. \quad (5)$$

25 Scalar curvature is  $R = R_{\mu\nu}g^{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$  and  $\square h(t) = -\partial_t^2 h(t) - 3H\partial_t h(t)$ , where  $H = \frac{\dot{a}}{a}$  is the Hubble parameter.

**Lemma 2.1.** *If the metric is chosen to be FLRW, then the system (2) has two linearly independent equations.*

*Proof.* Note, that for the functions that only depend on time we have  $K_{\mu\nu} = 0$  when  $\mu \neq \nu$ . Also  $\Omega_{pq\mu\nu} = 0$  for  $\mu \neq \nu$ . It means that system (2) has four nontrivial equations. Equations with indices  $\mu\nu$  equal to 11, 22 and 33 can be rewritten as

$$\begin{aligned} & 16\pi G g_{ii} \left( -\frac{1}{2} R^p \mathcal{F}(\square) R^q + \left( \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 \right) W_{pq} - \left( \ddot{W}_{pq} + 2 \frac{\dot{a}}{a} \dot{W}_{pq} \right) \right. \\ & \left. + \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (\nabla^{\lambda} \square^l R^p \nabla_{\lambda} \square^{n-1-l} R^q + \square^l R^p \square^{n-l} R^q) \right) = g_{ii} \left( \frac{2\ddot{a}a + \dot{a}^2}{a} - \Lambda \right). \end{aligned} \quad (6)$$

These equations are clearly proportional to each other and thus we have altogether two independent equations. The most convenient choice is to use trace and 00 equations, which are respectively

$$-2R^p \mathcal{F}(\square) R^q + R W_{pq} + 3 \square W_{pq} + \frac{1}{2} \Omega_{pq} = \frac{R - 4\Lambda}{16\pi G}, \quad (7)$$

$$\frac{1}{2} R^p \mathcal{F}(\square) R^q + R_{00} W_{pq} - K_{00} W_{pq} + \frac{1}{2} \Omega_{pq\,00} = -\frac{G_{00} - \Lambda}{16\pi G}, \quad (8)$$

$$\Omega_{pq} = g^{\mu\nu} \Omega_{pq\,\mu\nu}. \quad (9)$$

□

30 At first, we investigate how does equations (2) change when parameters  $p$  and  $q$  replace their places in the action (1). To this end, the following lemma holds.

**Lemma 2.2.** *If we consider actions  $S_{pq}$ , given in (1), and  $S_{qp}$ . The corresponding equations of motion are equivalent.*

*Proof.* Equations of motion, given by equation (2), for actions  $S_{pq}$  and  $S_{qp}$  read

$$-\frac{1}{2}g_{\mu\nu}R^p\mathcal{F}(\square)R^q + R_{\mu\nu}W_{pq} - K_{\mu\nu}W_{pq} + \frac{1}{2}\Omega_{pq\mu\nu} = -\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G}, \quad (10)$$

$$-\frac{1}{2}g_{\mu\nu}R^q\mathcal{F}(\square)R^p + R_{\mu\nu}W_{qp} - K_{\mu\nu}W_{qp} + \frac{1}{2}\Omega_{qp\mu\nu} = -\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G}. \quad (11)$$

Since  $W_{pq}$  and  $W_{qp}$  coincide, subtraction of the last two equation yields

$$(R^p\mathcal{F}(\square)R^q - R^q\mathcal{F}(\square)R^p) = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (\square^l R^p \square^{n-l} R^q - \square^l R^q \square^{n-l} R^p). \quad (12)$$

The terms obtained for  $l = 0$  on the right hand side of the last equation exactly match the left hand side, thus we are left with

$$\sum_{n=1}^{\infty} f_n \sum_{l=1}^{n-1} (\square^l R^p \square^{n-l} R^q - \square^l R^q \square^{n-l} R^p) = 0. \quad (13)$$

35 This equation can be proved by changing the summation index  $l \rightarrow n-l$  in one of the terms, which completes the proof. It is worth noting that we assume that total derivatives terms arising from the partial integration vanish.  $\square$

### 3. The scale factor

In this paper we consider the scale factor in the form

$$a(t) = Ce^{-\frac{\gamma}{12}t^2}. \quad (14)$$

This form of scale factor has been introduced, as a solution of the  $p = q = 1$  case in the paper [7]. As a solution of  $R^2$  gravity the scale factor (14) has been  
40 introduced in [13] and further studied in papers [14, 15]. The correspondence between the  $R^2$  gravity model and nonlocal model for  $p = q = 1$  has been discussed in [16]. The present paper generalizes this result to various values of parameters  $p$  and  $q$  in action  $S_{pq}$ . It is worth noting that  $\gamma = 0$  gives the  
45 Minkowski spacetime, and it is a solution of equations of motion (2) for  $\Lambda = 0$ .

The following analysis does not depend on the sign of  $\gamma$  and gives expanding ( $\gamma < 0$ ) and contracting ( $\gamma > 0$ ) models.

The Hubble parameter and scalar curvature are linear and quadratic functions in cosmic time  $t$ , respectively

$$H(t) = -\frac{1}{6}\gamma t, \quad R(t) = \frac{1}{3}\gamma(\gamma t^2 - 3). \quad (15)$$

By direct calculation one can show that for any natural number  $p$ ,  $\square R^p$  is a linear combination of  $R^p$ ,  $R^{p-1}$  and  $R^{p-2}$ , i.e.

$$\square R^p = p\gamma R^p - \frac{p}{3}(4p-5)\gamma^2 R^{p-1} - \frac{4}{3}p(p-1)\gamma^3 R^{p-2}. \quad (16)$$

**Lemma 3.1.** *For fixed value of parameter  $\gamma$ , and therefore fixed values of Hubble parameter  $H$  and scalar curvature  $R$ , consider the space  $P_p(R)$  of all polynomials of degree at most  $p$  in  $R$  and its base  $v_p = \begin{pmatrix} R^p & R^{p-1} & \dots & R & 1 \end{pmatrix}^T$ . Operator  $\square$  is a linear operator on  $P_p(R)$ . The matrix of the operator  $\square$  in the basis  $v_p$  is*

$$M_p = \gamma \begin{pmatrix} p & \frac{p}{3}(5-4p)\gamma & \frac{4}{3}p(1-p)\gamma^2 & 0 & \dots & 0 \\ 0 & p-1 & \frac{p-1}{3}(9-4p)\gamma & \frac{4}{3}(1-p)(p-2)\gamma^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & \frac{\gamma}{3} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (17)$$

*Proof.* Since  $\square h(t) = -\partial_t^2 h(t) - 3H\partial_t h(t)$  it is clear that  $\square$  is a linear operator.

It remains to prove that  $\square R^s \in P_p(R)$  for all  $0 \leq s \leq p$ . For  $s = 0$ , we have  $\square 1 = 0 \in P_p(R)$ . For  $s = 1$  equation (16) becomes  $\square R = \gamma R + \frac{\gamma^2}{3}$  which is a linear polynomial in  $R$  and therefore an element of  $P_p(R)$ . For  $2 \leq s \leq p$  equation (16) gives us that  $\square R^s$  is a polynomial of degree  $s$  in  $R$  and hence element of  $P_p(R)$ . Again, from equation (16) we obtain that the matrix  $M_p$  has the form given in lemma.  $\square$

As a consequence of the lemma 3.1  $\square^n R^p$  is expressible as a polynomial in  $R$  of degree  $p$ . Let  $F_p$  be the matrix of the operator  $\mathcal{F}(\square)$ ,

$$F_p = \sum_{n=0}^{\infty} f_n M_p^n = \mathcal{F}(M_p). \quad (18)$$

55 **4. The general case**

Lemma 2.2 allow us to assume that  $p \geq q$  in the following sections. Also, it is worth noting that  $W_{pq}$  is a polynomial of degree  $p + q - 1$  in  $R$ , as well as

$$\mathcal{F}(\square)R^p = e_p F_p v_p, \quad (19)$$

$$W_{pq} = pR^{p-1}e_q F_q v_q + qR^{q-1}e_p F_p v_p, \quad (20)$$

$$\square W_{pq} = -\frac{4}{3}\gamma^2(R + \gamma)W''_{pq} - \frac{2}{3}\gamma^2 W'_{pq}, \quad (21)$$

$$K_{00}W_{pq} = \gamma(R + \gamma)W'_{pq}, \quad (22)$$

and  $'$  denotes derivation *wrt*  $R$ . The  $\Omega_{pq}$  and  $\Omega_{pq\ 00}$  terms are also polynomials in  $R$  with degrees  $p + q - 1$  and  $p + q$  respectively

$$\Omega_{pq} = -2S_1 + 4S_2, \quad (23)$$

$$\Omega_{pq\ 00} = -S_1 - S_2, \quad (24)$$

$$S_1 = \frac{4}{3}\gamma^2(R + \gamma) \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} e_p M_p^l D_p v_p e_q M_q^{n-1-l} D_q v_q, \quad (25)$$

$$S_2 = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} e_p M_p^l v_p e_q M_q^{n-l} v_q. \quad (26)$$

Hence, equations (7) and (8) are polynomial type in  $R$  and their degree is  $p + q$ . Let us look at the highest order coefficient. If  $p \neq q$  we obtain the following two equations

$$p\mathcal{F}(q\gamma)(q - p + 2) + q\mathcal{F}(p\gamma)(q - p - 2) = 0, \quad (27)$$

$$(-q - \frac{1}{2}p(q - p))\mathcal{F}(q\gamma) + (-\frac{1}{2}q(q - p) + q)\mathcal{F}(p\gamma) = 0. \quad (28)$$

These equations are linearly dependent for any values of parameters  $p \neq q$ .

In the other case  $p = q$  the previous system becomes

$$(p - 1)\mathcal{F}(p\gamma) + p\gamma\mathcal{F}'(p\gamma) = 0, \quad (29)$$

$$-\frac{1}{2}(p - 1)\mathcal{F}(p\gamma) - \frac{1}{2}p\gamma\mathcal{F}'(p\gamma) = 0. \quad (30)$$

We conclude that, this case also yields linearly dependent equations.

## 5. Particular cases

### 5.1. Case $p = 1, q = 1$

At the beginning we discuss the simplest case  $p = q = 1$ . The solution of the form  $a(t) = C \exp(\Lambda t^2)$  was already obtained by Koshelev and Vernov in [7].

**Theorem 5.1.** *If the scale factor has the form  $a(t) = C e^{-\frac{\gamma}{12} t^2}$  and  $p = q = 1$ , then the system (7), (8) is satisfied iff  $\gamma = -12\Lambda$ ,  $\mathcal{F}'(\gamma) = 0$  and  $f_0 = \frac{3\kappa}{2\gamma} - 8\mathcal{F}(\gamma)$  where  $\kappa = \frac{1}{16\pi G}$ .*

*Proof.* Trace and 00 equations are written as

$$T_2 R^2 + T_1 R + T_0 = 0, \quad (31)$$

$$Z_2 R^2 + Z_1 R + Z_0 = 0, \quad (32)$$

where

$$\begin{aligned} T_0 &= \frac{2}{9} (-5\mathcal{F}'(\gamma)\gamma^3 + 8\mathcal{F}(\gamma)\gamma^2 + f_0\gamma^2 + 18\kappa\Lambda), \\ T_1 &= \frac{1}{3} (-3\kappa + 16\gamma\mathcal{F}(\gamma) + 2\gamma f_0), \\ T_2 &= 2\gamma\mathcal{F}'(\gamma), \\ Z_0 &= \frac{1}{36} (-26\mathcal{F}'(\gamma)\gamma^3 - 64\mathcal{F}(\gamma)\gamma^2 - 8f_0\gamma^2 + 9\kappa\gamma - 36\kappa\Lambda), \\ Z_1 &= \frac{1}{12} (-12\mathcal{F}'(\gamma)\gamma^2 - 16\mathcal{F}(\gamma)\gamma - 2f_0\gamma + 3\kappa), \\ Z_2 &= -\frac{1}{2}\gamma\mathcal{F}'(\gamma). \end{aligned} \quad (33)$$

At first, note that  $T_0 + 4Z_0 = 4\gamma Z_1$ ,  $T_1 + 4Z_1 = 8\gamma Z_2$ ,  $T_2 + 4Z_2 = 0$  and hence equations (31) and (32) are equivalent. Therefore it is sufficient only to look at the trace. On the other hand from equation (15) we see that  $R$  is a quadratic function in time and hence equation (31) is satisfied for all values of time  $t$  iff  $T_0 = T_1 = T_2 = 0$ . yields the linear system in  $f_0$ ,  $\mathcal{F}(\gamma)$  and  $\mathcal{F}'(\gamma)$ . It is consistent only for  $\gamma = -12\Lambda$  and the solution is

$$\mathcal{F}'(\gamma) = 0, \quad f_0 = \frac{3\kappa}{2\gamma} - 8\mathcal{F}(\gamma). \quad (34)$$

□

### 5.2. Case $(p, q) \neq (1, 1)$

As a consequence of lemma 3.1 we can write

$$R^p = e_p v_p, \quad \square^n R^p = e_p M_p^n v_p, \quad \mathcal{F}(\square) R^p = e_p F_p v_p, \quad (35)$$

where  $e_p$  are the coordinates of  $R^p$  in basis  $v_p$  and  $D_p$  be a matrix such that  $\frac{\partial v_p}{\partial R} = D_p v_p$  and  $p \in \mathbb{N}$ . Therefore the system (7), (8) can be written as

$$\frac{R - 4\Lambda}{16\pi G} = RW_{pq} - 4\gamma^2(R + \gamma)W''_{pq} - 2\gamma^2 W'_{pq} - 2e_p v_p e_q F_q v_q - S_1 + 2S_2, \quad (36)$$

$$\frac{\Lambda - G_{00}}{16\pi G} = \frac{1}{2}e_p v_p e_q F_q v_q + \frac{\gamma}{4}(\gamma - R)W_{pq} - \gamma(R + \gamma)W'_{pq} - \frac{1}{2}(S_1 + S_2), \quad (37)$$

where

$$W_{pq} = e_p D_p v_p e_q F_q v_q + e_q D_q v_q e_p F_p v_p, \quad (38)$$

$$S_1 = \frac{4}{3}\gamma^2(R + \gamma) \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} e_p M_p^l D_p v_p e_q M_q^{n-1-l} D_q v_q, \quad (39)$$

$$S_2 = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} e_p M_p^l v_p e_q M_q^{n-l} v_q. \quad (40)$$

**Theorem 5.2.** *Let*

$$T = -2e_p v_p e_q F_q v_q + RW_{pq} - 4\gamma^2(R + \gamma)W''_{pq} - 2\gamma^2 W'_{pq} - S_1 + 2S_2 - \frac{R - 4\Lambda}{16\pi G}, \quad (41)$$

$$Z = \frac{1}{2}e_p v_p e_q F_q v_q + \frac{\gamma}{4}(\gamma - R)W_{pq} - \gamma(R + \gamma)W'_{pq} - \frac{1}{2}(S_1 + S_2) + \frac{G_{00} - \Lambda}{16\pi G}, \quad (42)$$

then  $T + 4Z = 4\gamma Z'$ . The equations (36) and (37) are equivalent.

*Proof.* To prove the first part of the theorem, direct calculations give

$$T + 4Z = \gamma W_{pq} - \gamma(R + 3\gamma)W'_{pq} - 4\gamma^2(R + \gamma)W''_{pq} - 3S_1, \quad (43)$$

$$4\gamma Z' = 2\gamma(R^p \mathcal{F}(\square) R^q)' - \gamma W_{pq} - \gamma(R + 3\gamma)W'_{pq} - 4\gamma^2(R + \gamma)W''_{pq} - 2\gamma(S_1 + S_2)'. \quad (44)$$



After some simplifications we are left with the following equation

$$W_{pq} - (R^p \mathcal{F}(\square) R^q)' = \frac{3}{2} \gamma^{-1} S_1 - S'_1 - S'_2. \quad (45)$$

Instead of expressing  $\square^n R^p$  in basis  $v_p$ , and  $\square^n R^q$  in basis  $v_q$  it is more convenient to express all the terms in basis  $v_{p+q}$ . Therefore let  $\varepsilon_p$  and  $\varepsilon_q$  be a coordinates of  $R^p$  and  $R^q$  respectively in basis  $v_{p+q}$ . Then the above equation becomes

$$\sum_{n=1}^{\infty} f_n \varepsilon_p (M_{p+q}^n v_{p+q} \varepsilon_q - v_{p+q} \varepsilon_q M_{p+q}^n) D_{p+q} v_{p+q} = \sum_{n=1}^{\infty} f_n q_n \quad (46)$$

where

$$\begin{aligned} q_n = \sum_{l=0}^{n-1} & \left( 2\gamma(R + \frac{\gamma}{3}) \varepsilon_p M_{p+q}^l D_{p+q} v_{p+q} \varepsilon_q M_{p+q}^{n-l-1} D_{p+q} v_{p+q} \right. \\ & - \frac{4}{3} \gamma^2 (R + \gamma) \varepsilon_p M_{p+q}^l D_{p+q}^2 v_{p+q} \varepsilon_q M_{p+q}^{n-l-1} D_{p+q} v_{p+q} \\ & - \frac{4}{3} \gamma^2 (R + \gamma) \varepsilon_p M_{p+q}^l D_{p+q} v_{p+q} \varepsilon_q M_{p+q}^{n-l-1} D_{p+q}^2 v_{p+q} \\ & - \varepsilon_p M_{p+q}^l D_{p+q} v_{p+q} \varepsilon_q M_{p+q}^{n-l} v_{p+q} \\ & \left. - \varepsilon_p M_{p+q}^l v_{p+q} \varepsilon_q M_{p+q}^{n-l} D_{p+q} v_{p+q} \right). \end{aligned} \quad (47)$$

It is sufficient to prove

$$M_{p+q}^n v_{p+q} \varepsilon_q - v_{p+q} \varepsilon_q M_{p+q}^n = q_n. \quad (48)$$

Now, move the last term in  $q_n$  to the left side and after some index relabeling one obtains

$$\begin{aligned} & \sum_{l=0}^{n-1} \varepsilon_p M_{p+q}^{l+1} v_{p+q} \varepsilon_q M_{p+q}^{n-l-1} D_{p+q} v_{p+q} \\ & = \sum_{l=0}^{n-1} \left( 2\gamma(R + \frac{\gamma}{3}) \varepsilon_p M_{p+q}^l D_{p+q} v_{p+q} \varepsilon_q M_{p+q}^{n-l-1} D_{p+q} v_{p+q} \right. \\ & - \frac{4}{3} \gamma^2 (R + \gamma) \varepsilon_p M_{p+q}^l D_{p+q}^2 v_{p+q} \varepsilon_q M_{p+q}^{n-l-1} D_{p+q} v_{p+q} \\ & - \frac{4}{3} \gamma^2 (R + \gamma) \varepsilon_p M_{p+q}^l D_{p+q} v_{p+q} \varepsilon_q M_{p+q}^{n-l-1} D_{p+q}^2 v_{p+q} \\ & \left. - \varepsilon_p M_{p+q}^l D_{p+q} v_{p+q} \varepsilon_q M_{p+q}^{n-l} v_{p+q} \right). \end{aligned} \quad (49)$$

Let us introduce matrix function  $\alpha$  by :

$$\alpha(X) = \sum_{l=0}^{n-1} M_{p+q}^l X M_{p+q}^{n-l-1}. \quad (50)$$

Then the previous equation becomes

$$\begin{aligned} \varepsilon_p M_{p+q} \alpha(v_{p+q} \varepsilon_q) D_{p+q} v_{p+q} &= 2\gamma(R + \frac{\gamma}{3}) \varepsilon_p \alpha(D_{p+q} v_{p+q} \varepsilon_q) D_{p+q} v_{p+q} \\ &\quad - \frac{4}{3} \gamma^2 (R + \gamma) \varepsilon_p \alpha(D_{p+q}^2 v_{p+q} \varepsilon_q) D_{p+q} v_{p+q} \\ &\quad - \frac{4}{3} \gamma^2 (R + \gamma) \varepsilon_p \alpha(D_{p+q} v_{p+q} \varepsilon_q) D_{p+q}^2 v_{p+q} \\ &\quad - \varepsilon_p \alpha(D_{p+q} v_{p+q} \varepsilon_q) M_{p+q} v_{p+q}. \end{aligned} \quad (51)$$

Finally, the last equation is equivalent to

$$\begin{aligned} \varepsilon_p M_{p+q} \alpha(v_{p+q} \varepsilon_q) D_{p+q} v_{p+q} &= \gamma(R + \frac{\gamma}{3}) \varepsilon_p \alpha(D_{p+q} v_{p+q} \varepsilon_q) D_{p+q} v_{p+q} \\ &\quad - \frac{4}{3} \gamma^2 (R + \gamma) \varepsilon_p \alpha(D_{p+q}^2 v_{p+q} \varepsilon_q) D_{p+q} v_{p+q}. \end{aligned} \quad (52)$$

$$\begin{aligned} \varepsilon_p \left( M_{p+q} \alpha(v_{p+q} \varepsilon_q) - \gamma(R + \frac{\gamma}{3}) \alpha(D_{p+q} v_{p+q} \varepsilon_q) \right. \\ \left. + \frac{4}{3} \gamma^2 (R + \gamma) \alpha(D_{p+q}^2 v_{p+q} \varepsilon_q) \right) D_{p+q} v_{p+q} = 0. \end{aligned} \quad (53)$$

Recall equation (21), which states

$$\square u = \gamma(R + \frac{\gamma}{3}) u' - \frac{4}{3} \gamma^2 (R + \gamma) u''. \quad (54)$$

Apply this equation to all elements of basis  $v_{p+q}$  and get

$$\begin{aligned} M_{p+q} v_{p+q} - \gamma(R + \frac{\gamma}{3}) v'_{p+q} + \frac{4}{3} \gamma^2 (R + \gamma) v''_{p+q} &= 0, \\ M_{p+q} v_{p+q} - \gamma(R + \frac{\gamma}{3}) D_{p+q} v_{p+q} + \frac{4}{3} \gamma^2 (R + \gamma) D_{p+q}^2 v_{p+q} &= 0. \end{aligned} \quad (55)$$

Multiplying by  $M_{p+q}^l$  from the left and  $\varepsilon_q M_{p+q}^{n-1-l}$  from the right and summing over  $l$  from 0 to  $n-1$  we get

$$\begin{aligned} M_{p+q} \alpha(v_{p+q} \varepsilon_q) - \gamma(R + \frac{\gamma}{3}) \alpha(D_{p+q} v_{p+q} \varepsilon_q) \\ + \frac{4}{3} \gamma^2 (R + \gamma) \alpha(D_{p+q}^2 v_{p+q} \varepsilon_q) = 0. \end{aligned} \quad (56)$$

At the end, multiplying by  $\varepsilon_p$  from the left and  $D_{p+q} v_{p+q}$  from the right com-

70 pletes the proof of the first part of the theorem.

As we have seen previously,  $T$  and  $Z$  are polynomials in  $R$  of degree  $p + q$ . Let their coefficients be  $T_j$  and  $Z_j$  ( $0 \leq j \leq p + q$ ) respectively. What we have proved so far implies

$$\begin{aligned} T_{p+q} + 4Z_{p+q} &= 0, \\ T_j + 4Z_j &= 4\gamma(j+1)Z_{j+1}, (j \leq 0 < p+q). \end{aligned} \tag{57}$$

Moreover, the above equations imply that the systems  $T_{p+q} = T_{p+q-1} = \dots = T_0 = 0$  and  $Z_{p+q} = Z_{p+q-1} = \dots = Z_0 = 0$  are equivalent, i.e. equations (36) and (37) are equivalent.  $\square$

It remains to solve equation (36). That is a very difficult task and it can be  
 75 done only for particular values of parameters  $p$  and  $q$ . The following theorem gives three cases. All other possibilities for  $q \leq p \leq 4$  are presented in appendix.

**Theorem 5.3.** *The equation (36) is satisfied in the following cases ( $\kappa = \frac{1}{16\pi G}$ ):*

$$\begin{aligned} p = 2, q = 1: & \mathcal{F}(\gamma) = \frac{9\kappa(\gamma+9\Lambda)}{112\gamma^3}, \mathcal{F}(2\gamma) = \frac{3\kappa(\gamma+9\Lambda)}{56\gamma^3}, f_0 = -\frac{\kappa(4\gamma+15\Lambda)}{7\gamma^3}, \mathcal{F}'(\gamma) = -\frac{3\kappa(\gamma+9\Lambda)}{8\gamma^4}, \\ p = 2, q = 2: & \mathcal{F}(\gamma) = \frac{369\kappa(\gamma+8\Lambda)}{9344\gamma^4}, \mathcal{F}(2\gamma) = \frac{27\kappa(\gamma+8\Lambda)}{4672\gamma^4}, f_0 = \frac{\kappa(145\gamma+576\Lambda)}{876\gamma^4}, \mathcal{F}'(\gamma) = \\ 80 & -\frac{639\kappa(\gamma+8\Lambda)}{2336\gamma^5}, \mathcal{F}'(2\gamma) = -\frac{27\kappa(\gamma+8\Lambda)}{9344\gamma^5}, \\ p = 3, q = 1: & \mathcal{F}(\gamma) = \frac{\kappa(107\gamma+408\Lambda)}{6432\gamma^4}, \mathcal{F}(2\gamma) = -\frac{\kappa(173\gamma+840\Lambda)}{7504\gamma^4}, \mathcal{F}(3\gamma) = 0, f_0 = -\frac{\kappa(95\gamma+768\Lambda)}{268\gamma^4}, \\ & \mathcal{F}'(\gamma) = -\frac{9\kappa(\gamma+8\Lambda)}{88\gamma^5}. \end{aligned}$$

*Proof.* One can see that, for each of the values  $p$  and  $q$  listed in the theorem, all of the coefficients  $T_j$  are linear combinations of the following  $p+q+1$  "variables"  
 85  $f_0 = \mathcal{F}(0), \mathcal{F}(\gamma), \dots, \mathcal{F}(p\gamma), \mathcal{F}'(\gamma), \dots, \mathcal{F}'(q\gamma)$ . Hence trace equation (7) and 00 equations are split into linear systems of  $p+q+1$  equations with  $p+q+1$  variables. Equation (57) implies that these two systems are equivalent. Solving one of them (for example trace) in each of the three cases gives the statement of the theorem.

In particular, for  $p = 2$  and  $q = 1$  the coefficients  $T_j$  are given by:

$$\begin{aligned}
T_0 &= \frac{4}{9} (-5\mathcal{F}'(\gamma)\gamma^4 - 32\mathcal{F}(\gamma)\gamma^3 - 19\mathcal{F}(2\gamma)\gamma^3 - 3f_0\gamma^3 + 9\kappa\Lambda), \\
T_1 &= \frac{1}{3} (10\mathcal{F}(\gamma)\gamma^2 - 55\mathcal{F}(2\gamma)\gamma^2 - 9f_0\gamma^2 - 3\kappa), \\
T_2 &= 2\gamma (10\mathcal{F}(\gamma) - \mathcal{F}(2\gamma) + 2\gamma\mathcal{F}'(\gamma)), \\
T_3 &= 3\mathcal{F}(2\gamma) - 2\mathcal{F}(\gamma),
\end{aligned} \tag{58}$$

The system that remains to be solved is

$$\begin{aligned}
-5\mathcal{F}'(\gamma)\gamma^4 - 32\mathcal{F}(\gamma)\gamma^3 - 19\mathcal{F}(2\gamma)\gamma^3 - 3f_0\gamma^3 &= -9\kappa\Lambda, \\
10\mathcal{F}(\gamma)\gamma^2 - 55\mathcal{F}(2\gamma)\gamma^2 - 9f_0\gamma^2 &= 3\kappa, \\
10\mathcal{F}(\gamma) - \mathcal{F}(2\gamma) + 2\gamma\mathcal{F}'(\gamma) &= 0, \\
3\mathcal{F}(2\gamma) - 2\mathcal{F}(\gamma) &= 0.
\end{aligned} \tag{59}$$

The solution of this system is given in the theorem. The other cases are proved in the similar way.

□

## 6. Limit $M \rightarrow +\infty$

Let the characteristic scale  $M$  grow to infinity, the function  $\mathcal{F}(\frac{\square}{M^2}) \rightarrow f_0$  and hence the EOM (7), (8) yield only conditions on  $f_0$ . In case  $(p, q) \neq (1, 1)$  there is no solution. The case  $(p, q) = (1, 1)$  provides the following equations

$$18\gamma f_0 = 3\kappa, \quad 9\gamma^2 f_0 = -2\kappa\Lambda. \tag{60}$$

The solution of this system is  $f_0 = \frac{\kappa}{6\gamma}$  if the cosmological constant  $\Lambda$  is such that  $\Lambda = -\frac{\gamma}{12}$ . Taking the further assumption  $f_0 = 0$  restores the General Relativity and we see that scale factor (14) is not a solution.

## 7. Conclusion

In this paper we have presented cosmological bounce solution of the form  $a(t) = a_0 \exp(-\frac{\gamma}{12}t^2)$ . This solution is obtained in the modified gravity model

with nonlocal term  $R^p \mathcal{F}(\square) R^q$ . In order to have a solution analytic function  $\mathcal{F}$  and its derivative  $\mathcal{F}'$  have to satisfy conditions of the form

$$f_0 = x_0, \quad \mathcal{F}(k\gamma) = x_k \ (1 \leq k \leq p), \quad \mathcal{F}'(l\gamma) = y_l \ (1 \leq l \leq q), \quad (61)$$

for some constants  $x_k$  and  $y_l$ . It is worth noting that in all cases except  $p = q = 1$  the set of constants  $x_k$  and  $y_l$  is unique (for fixed values of  $\gamma$  and  $\Lambda$ ). Moreover, the case  $p = q = 1$  requires that constant  $\gamma$  has special value ( $\gamma = -12\Lambda$ ) which means that cosmological constant is required in order to have nontrivial solution. In the other cases there is no such restriction.

In the present paper we considered the model for particular values of the parameters  $p$  and  $q$ . There is a possibility to extend some of these results to a general case. Matrices  $M_p$  and  $F_p$  can be defined for negative integer values of  $p$ , but it is much harder to do the computations since they are infinitely dimensional, and it is not clear if it will yield any solutions. For example, the model  $p = -1, q = 1$  was considered in [17, 18].

## 8. Appendix

In a similar way as theorem 5.3 the following six cases are proved:

**Theorem 8.1.** *The equation (36) is satisfied in the following cases ( $\kappa = \frac{1}{16\pi G}$ ):*

$$\begin{aligned} p = 3, q = 2: \quad & \mathcal{F}(\gamma) = \frac{3\kappa(10702\gamma+40497\Lambda)}{245680\gamma^5}, \quad \mathcal{F}(2\gamma) = -\frac{27\kappa(6\gamma+25\Lambda)}{24568\gamma^5}, \quad \mathcal{F}(3\gamma) = -\frac{27\kappa(6\gamma+25\Lambda)}{49136\gamma^5}, \\ & f_0 = -\frac{3\kappa(7099\gamma+23949\Lambda)}{15355\gamma^5}, \quad \mathcal{F}'(\gamma) = -\frac{3\kappa(11614\gamma+68865\Lambda)}{270248\gamma^6}, \quad \mathcal{F}'(2\gamma) = \frac{513\kappa(6\gamma+25\Lambda)}{171976\gamma^6}, \\ p = 3, q = 3: \quad & \mathcal{F}(\gamma) = \frac{\kappa(338597\gamma+1847844\Lambda)}{9513152\gamma^6}, \quad \mathcal{F}(2\gamma) = -\frac{21\kappa(379\gamma+2076\Lambda)}{432416\gamma^6}, \quad \mathcal{F}(3\gamma) = -\frac{9\kappa(379\gamma+2076\Lambda)}{864832\gamma^6}, \\ & f_0 = \frac{9\kappa(77093\gamma+441108\Lambda)}{1405352\gamma^6}, \quad \mathcal{F}'(\gamma) = -\frac{3\kappa(1462285\gamma+8126148\Lambda)}{13080584\gamma^7}, \quad \mathcal{F}'(2\gamma) = \frac{255\kappa(379\gamma+2076\Lambda)}{1324274\gamma^7}, \\ & \mathcal{F}'(3\gamma) = \frac{3\kappa(379\gamma+2076\Lambda)}{432416\gamma^7}, \\ p = 4, q = 1: \quad & \mathcal{F}(\gamma) = \frac{3\kappa(1570\gamma+11679\Lambda)}{1800928\gamma^5}, \quad \mathcal{F}(2\gamma) = -\frac{9\kappa(50102\gamma+262581\Lambda)}{80141296\gamma^5}, \quad \mathcal{F}(3\gamma) = -\frac{3\kappa(105430\gamma+726207\Lambda)}{49525520\gamma^5}, \\ & \mathcal{F}(4\gamma) = -\frac{3\kappa(1570\gamma+11679\Lambda)}{2251160\gamma^5}, \quad f_0 = -\frac{3\kappa(1111\gamma+5361\Lambda)}{56279\gamma^5}, \quad \mathcal{F}'(\gamma) = -\frac{27\kappa(2\gamma+15\Lambda)}{2320\gamma^6}, \\ p = 4, q = 2: \quad & \mathcal{F}(\gamma) = \frac{27\kappa(116489\gamma+976692\Lambda)}{2128403200\gamma^6}, \quad \mathcal{F}(2\gamma) = -\frac{27\kappa(4591\gamma+19308\Lambda)}{170272256\gamma^6}, \quad \mathcal{F}(3\gamma) = \frac{9\kappa(2632969\gamma+9126132\Lambda)}{93649740800\gamma^6}, \\ & \mathcal{F}(4\gamma) = 0, \quad f_0 = \frac{27\kappa(9773\gamma+38204\Lambda)}{6651260\gamma^6}, \quad \mathcal{F}'(\gamma) = -\frac{9\kappa(36889711\gamma+230208108\Lambda)}{22044176000\gamma^7}, \\ & \mathcal{F}'(2\gamma) = \frac{9\kappa(1257961\gamma-26340492\Lambda)}{108244505600\gamma^7}, \end{aligned}$$

$$\begin{aligned}
p = 4, q = 3: \mathcal{F}(\gamma) &= \frac{\kappa(21007019473\gamma + 144494046423\Lambda)}{2412863869280\gamma^7}, \mathcal{F}(2\gamma) = -\frac{3\kappa(1426277827\gamma + 10110884265\Lambda)}{1206431934640\gamma^7}, \\
\mathcal{F}(3\gamma) &= \frac{9\kappa(32585957\gamma + 237505338\Lambda)}{4825727738560\gamma^7}, \mathcal{F}(4\gamma) = \frac{\kappa(32585957\gamma + 237505338\Lambda)}{1206431934640\gamma^7}, \\
125 \quad f_0 &= \frac{3\kappa(3321165266\gamma + 25006112775\Lambda)}{75401995915\gamma^7}, \mathcal{F}'(\gamma) = -\frac{3\kappa(43005362079625\gamma + 307070903674071\Lambda)}{1924258935750800\gamma^8}, \\
\mathcal{F}'(2\gamma) &= \frac{\kappa(15505640343740\gamma + 110842995690981\Lambda)}{1503214190561440\gamma^8}, \mathcal{F}'(3\gamma) = -\frac{109\kappa(32585957\gamma + 237505338\Lambda)}{26541502562080\gamma^8}, \\
p = 4, q = 4: \mathcal{F}(\gamma) &= \frac{3\kappa(37038228809\gamma + 146181469392\Lambda)}{127137036761600\gamma^8}, \mathcal{F}(2\gamma) = -\frac{9\kappa(238071667\gamma + 847503216\Lambda)}{7803583635712\gamma^8}, \\
\mathcal{F}(3\gamma) &= \frac{537\kappa(765701\gamma + 2682288\Lambda)}{9644878650880\gamma^8}, \mathcal{F}(4\gamma) = \frac{3\kappa(765701\gamma + 2682288\Lambda)}{219201787520\gamma^8}, f_0 = \frac{33\kappa(131820287\gamma + 420903432\Lambda)}{343872804172\gamma^8}, \\
\mathcal{F}'(\gamma) &= -\frac{81\kappa(261799491587\gamma + 967343633136\Lambda)}{4608717582608000\gamma^9}, \mathcal{F}'(2\gamma) = \frac{3\kappa(1392867522289\gamma + 4562593829712\Lambda)}{6945189435783680\gamma^9}, \\
130 \quad \mathcal{F}'(3\gamma) &= -\frac{4569\kappa(765701\gamma + 2682288\Lambda)}{26523416289920\gamma^9}, \mathcal{F}'(4\gamma) = -\frac{9\kappa(765701\gamma + 2682288\Lambda)}{876807150080\gamma^9}.
\end{aligned}$$

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